A Fuzzy Confidence Interval to test equality of means. Application based on the survey of Health, Ageing and Retirement in Europe (SHARE)

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ABSTRACT

In the context of fuzzy data analysis, we developed the methodology to construct fuzzy confidence intervals by the so-called technique of the likelihood ratio. In particular, the distribution of the likelihood ratio is estimated by a proper bootstrap algorithm, such that the randomly drawn observations will preserve the location and dispersion measures of the original fuzzy data set. Such intervals are suitable tools to test parameters. In particular, we show how to implement a hypothesis test for the equality of means of two groups, and we provide the decision rule. Our strategy will be to construct fuzzy confidence intervals for each parameter and then to analyse the overlapping area existing between them. We describe the method of construction of these particular intervals briefly. We recall the relative weight of the randomness vs fuzziness appearing in the process. Then, we explain how to use these intervals to test the equality of means. The decision rule will help us to reject or not the null hypothesis. We intend to show the practicability of our approach with an empirical application based on the survey of Health, Ageing and Retirement in Europe (SHARE Data).

Keywords: Fuzzy Statistics, Fuzzy Confidence Intervals, Testing equality of means, SHARE Data, Package R FuzzySTs

1. Introduction and motivation

The approach of considering the data as fuzzy, depending on the case, is gaining more and more importance. In fact, at the age of AI, it is imperative to know how to treat, i.e. to deal with, e.g. the perception of human reasoning appearing directly or not in many data. In such situations, vagueness and imprecision are inherent to the data and, consequently, one has to analyse them as fuzzy data. Fuzzy logic, initiated by L. Zadeh, has provided the indispensable tools to manage these situations. Keeping going and using the concepts of these new, non binary logic, fuzzy statistics open a broad field of new researches. Though much has been done in this fascinating area, limitations and difficulties arise as new techniques are developed. In particular, though fuzzy distributions could be postulated, inferential tools cannot straightforward be derived and used, and are more or less limited.

We intend in this study to tackle the problem of testing the equality of means in a framework of fuzzy data. Our fuzzy inference analysis will rest on fuzzy confidence intervals. We recently show how to construct such intervals by the so-called technique of the likelihood ratio (Berkachy and Donzé (2022)). Thanks to a bootstrap method, we are able to derive a proper distribution of this ratio and build the fuzzy interval. Berkachy (2021) and Berkachy and Donzé (2020a) show how to use these intervals to test hypotheses on individual parameters, for instance the testing of the fuzzy mean of a distribution. Berkachy and Donzé (2022) particularly show how to implement a hypothesis test for the equality of means of two groups, and give a practical decision rule. Our strategy will be to construct fuzzy confidence intervals for each parameter and then to analyse the overlapping area existing between them. Thereafter, we describe the method of construction of these particular intervals briefly. We recall the relative weight of the randomness vs fuzziness appearing in the process. Then, we explain how to use these intervals to test the equality of means. The decision rule will help us to reject or not the null hypothesis.

We present an empirical application based on the survey of Health, Ageing and Retirement in Europe. This panel dataset is well-known. Many studies have already been conducted using these data. In particular, this paper uses data from the generated easySHARE data set (Börsch-Supan and S. Gruber (2022)). See Stefan Gruber, Hunkler, and Stuck (2014) for methodological details. [1]

2. Fuzzy confidence intervals

2.1 Notation

Let us define by $\tilde{x}$ a fuzzy number. We write by $\mu_{\tilde{x}}(\cdot)$, the membership function. We consider also the α-cuts of $\tilde{x}$ denoted by $\tilde{x}_\alpha$ or by its equivalent in interval form by $[\tilde{x}^L_\alpha, \tilde{x}^R_\alpha]$. In practice, triangular or trapezoidal fuzzy numbers are often used. We denote them by a triplet $\tilde{x} = (p, q, r)$ respectively by $\tilde{x} = (p, q, r, s)$, with $p \leq q \leq r \leq s \in \mathbb{R}$.

2.2 Fuzzy confidence intervals

We first describe the construction of the traditional fuzzy confidence interval. Let be $X_1, \ldots, X_n$, a random sample with its corresponding fuzzy perception $\tilde{X}_1, \ldots, \tilde{X}_n$ (*in an epistemic approach*). The fuzzy realisations are $\tilde{x}_i$, $i = 1, \ldots, n$. We note the fuzzy random sample by $\tilde{\boldsymbol{X}}$. We assume that the distribution of X_i depends on a parameter θ about which we intend to do inferences. Using the fuzzy data, we can perform the test $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$ by first constructing a fuzzy confidence interval for θ at a significance level δ. Kruse and Meyer (1987) define a two-sided fuzzy confidence interval $\tilde{\Pi}$ for θ as:

Definition 2.1 (Fuzzy confidence interval (Kruse and Meyer (1987))).
Let $[\pi_1, \pi_2]$ be a symmetrical confidence interval for θ at the significance level δ. A fuzzy confidence interval $\tilde{\Pi}$ is a convex and normal fuzzy set such that its left and right α-cuts, respectively written by $\tilde{\Pi}_\alpha = [\tilde{\Pi}^L_\alpha, \tilde{\Pi}^R_\alpha]$, are written in the following manner:

$$\tilde{\Pi}^L_\alpha = \inf\big\{a \in \mathbb{R} : \exists x_i \in (\tilde{X}_i)_\alpha, \forall i = 1, \ldots, n, \text{ such that } \pi_1(x_1, \ldots, x_n) \leq a\big\},$$
$$\tilde{\Pi}^R_\alpha = \sup\big\{a \in \mathbb{R} : \exists x_i \in (\tilde{X}_i)_\alpha, \forall i = 1, \ldots, n, \text{ such that } \pi_2(x_1, \ldots, x_n) \geq a\big\}.$$

In Berkachy and Donzé (2020a, 2022), we generalise the traditional construction procedure of fuzzy confidence intervals by applying the concept of likelihood ratio. Note that this is not

[1]The easySHARE release 8.8.0 is based on SHARE Waves 1, 2, 3, 4, 5, 6, 7 and 8 (DOIs: 10.6103/SHARE.w1.800, 10.6103/SHARE.w2.800, 10.6103/SHARE.w3.800, 10.6103/SHARE.w4.800, 10.6103/SHARE.w5.800, 10.6103/SHARE.w6.800, 10.6103/SHARE.w7.800, 10.6103/SHARE.w8.800)

something new. See for example Gil and Casals (1988). The complete procedure we developed can be found in Berkachy and Donzé (2022). Let us in the following summarise the main concepts and steps. Based on Zadeh's probability concepts (Zadeh (1968)), the likelihood function of a fuzzy observation can be defined as:

Definition 2.2 (Likelihood function of a fuzzy observation)**.**

Let $\tilde{\theta}$ be a fuzzy parameter, possibly a vector, in the parameter space Θ. For a single fuzzy observation $\tilde{x}_i$, the likelihood function can be given by:

$$\mathcal{L}\tilde{\theta}; \tilde{x}_i) = P(\tilde{x}_i; \tilde{\theta}) = \int_{\mathbb{R}} \mu_{\tilde{x}_i}(x) f(x; \tilde{\theta}) \mathrm{d}x. \tag{1}$$

This probability can also be written using the α-cuts of the involved fuzzy numbers.

Let us now write the likelihood function for the fuzzy random sample $\tilde{\boldsymbol{X}}$ with fuzzy realisations $\tilde{\boldsymbol{x}}$:

$$\mathcal{L}(\tilde{\theta}; \tilde{\boldsymbol{x}}) = P(\tilde{\boldsymbol{x}}; \tilde{\theta}) = \int_{\mathbb{R}} \mu_{\tilde{x}_1} f(x; \tilde{\theta}) \mathrm{d}x \cdots \int_{\mathbb{R}} \mu_{\tilde{x}_n} f(x; \tilde{\theta}) \mathrm{d}x,$$

which in logarithms gives:

$$\ell(\tilde{\theta}; \tilde{\boldsymbol{x}}) = \ln(\mathcal{L}(\tilde{\theta}; \tilde{\boldsymbol{x}}) = \ln \int_{\mathbb{R}} \mu_{\tilde{x}_1} f(x; \tilde{\theta}) \mathrm{d}x + \ldots + \ln \int_{\mathbb{R}} \mu_{\tilde{x}_n} f(x; \tilde{\theta}) \mathrm{d}x.$$

The likelihood function is the following LR statistic:

$$\mathrm{LR} = -2 \ln \frac{\mathcal{L}(\tilde{\theta}; \tilde{\boldsymbol{x}})}{\mathcal{L}(\hat{\tilde{\theta}}_{\mathrm{ML}}; \tilde{\boldsymbol{x}})} = 2 \left[\ell(\hat{\tilde{\theta}}_{\mathrm{ML}}; \tilde{\boldsymbol{x}}) - \ell(\tilde{\theta}; \tilde{\boldsymbol{x}}) \right],$$

such that $\mathcal{L}(\tilde{\theta}; \tilde{\boldsymbol{x}}) \neq 0$ and $\mathcal{L}(\hat{\tilde{\theta}}_{\mathrm{ML}}; \tilde{\boldsymbol{x}}) \neq 0$, are both finite, and where $\hat{\tilde{\theta}}_{\mathrm{ML}}$ is the maximum likelihood estimator of the fuzzy parameter $\tilde{\theta}$.

As we cannot simply assume an asymptotically χ^2-distribution for LR, the question of its distribution arises naturally. We solve this problem by applying a specific bootstrap procedure (see Berkachy and Donzé (2022)).

Let be η the $(1-\delta)$-quantile of the distribution of the LR statistic. The confidence interval can be found as:

$$2 \left[\ell(\hat{\tilde{\theta}}_{\mathrm{ML}}; \tilde{\boldsymbol{x}}) - \ell(\tilde{\theta}; \tilde{\boldsymbol{x}}) \right] \leq \eta,$$

which gives:

$$\ell(\tilde{\theta}; \tilde{\boldsymbol{x}}) \geq \ell(\hat{\tilde{\theta}}_{\mathrm{ML}}; \tilde{\boldsymbol{x}}) - \frac{\eta}{2}.$$

Thus, the constructed interval is composed of all possible values $\tilde{\theta}$, for which the log-likelihood maximum varies by $\eta/2$ at most. A mandatory condition is that for every value of the parameter θ, the fuzzy confidence interval by the likelihood ratio $\tilde{\Pi}_{\mathrm{LR}}$ has to verify the following equation:

$$P\Big((\tilde{\Pi}_{\mathrm{LR}})^L_\alpha \leq \theta \leq (\tilde{\Pi}_{\mathrm{LR}})^R_\alpha \Big) \geq 1 - \delta, \quad \forall \alpha \in [0; 1].$$

We propose an ad hoc procedure to guarantee this condition (see Berkachy and Donzé (2022)). The R package `FuzzySTs` (Berkachy and Donzé (2020b)) gives the tools to build the fuzzy confidence interval $\tilde{\Pi}_{\mathrm{LR}}$.

2.3 Inference: comparison of means

Our aim is to compare the mean of two groups. Let us define the null hypothesis $\mathcal{H}_0$ that the means related to the two groups are equal, and the alternative one $\mathcal{H}_1$ that the pair of means is not equal. We wish thus to test at a significance level δ:

$$\mathcal{H}_0 : \mu_1 = \mu_2 \quad \text{against} \quad \mathcal{H}_1 : \mu_1 \neq \mu_2,$$

where μ_1 and μ_2 are the means of the groups 1 and 2 respectively. In a traditional approach, we would rewrite the test as a difference of parameters and effectively test this difference equal zero against the alternative. A translation of this approach in a fuzzy context is not so easy. First, the difference of fuzzy numbers are not so straightforward to compute and could lead to many problems. Second, a fuzzy distribution should be assumed for this difference, which could be difficult to justify. Thus, another way should be explored.

First of all, we construct for each group the fuzzy confidence intervals $\tilde{\Pi}_{\mathrm{LR}_1}$ and $\tilde{\Pi}_{\mathrm{LR}_2}$ by the above described method at the $1-\delta$ confidence level. We then analyse the overlapping between the intervals. The aim is to be able, using these intervals, to identify whether the means of groups are potentially equal or not. In case of perfect overlapping, we could infer that there is no difference between the means. We use the metric $d^{\theta^*}_{\mathrm{SGD}}$ proposed by Berkachy (2021) to measure the distance between the two fuzzy sets $\tilde{\Pi}_{\mathrm{LR}_1}$ and $\tilde{\Pi}_{\mathrm{LR}_2}$. See Berkachy and Donzé (2022) for a justification and a description of this particular distance.

Berkachy (2020) shows how to find a so-called optimal distance between two fuzzy sets, i.e. the position of the intervals such that both intervals become tangent. Let us denote this distance by $d^{\theta^*}_{\mathrm{SGD}}(\tilde{\Pi}_{\mathrm{LR}_1}, \tilde{\Pi}_{\mathrm{LR}_2})_{\mathrm{opt}}$. We are now able to propose the following R statistic given by:

$$\mathrm{R} = \frac{d^{\theta^*}_{\mathrm{SGD}}(\tilde{\Pi}_{\mathrm{LR}_1}, \tilde{\Pi}_{\mathrm{LR}_2})}{d^{\theta^*}_{\mathrm{SGD}}(\tilde{\Pi}_{\mathrm{LR}_1}, \tilde{\Pi}_{\mathrm{LR}_2})_{\mathrm{opt}}},$$

with $\mathrm{R} \in [0;1]$ by construction. This latter is helpful to make a decision. Indeed, we can formulate the following decision rule:

- The closer the R measure is to the value 0, the strongest we do not reject the null hypothesis $\mathcal{H}_0$;
- The closer the R measure is to the value 1, the strongest we reject the null hypothesis $\mathcal{H}_1$.

Note that our empirical experiences show us that already relatively small values of R, i.e. 0.05, suffice to reject the null hypothesis.

3. Fuzziness vs randomness

We provided in Berkachy and Donzé (2020a, 2022) simulation studies to explore the impact of the fuzziness of the parameters on the confidence intervals. It is not the fuzziness embodied naturally or after fuzzification in data, which is the centre of our attention. Indeed, following our procedure we do have to compute the fuzzy maximum likelihood parameter $\hat{\tilde{\theta}}_{\mathrm{ML}}$. For this purpose, we use the method proposed by Denoeux (2011). As the result of the latter is a crisp estimation, we thus have to fuzzify the parameter. Our simulations show that this step of fuzzification has a greater influence on the intervals than the randomness issues from the distribution of the data and appearing in the $(1-\delta)$-quantile of the distribution of the LR statistic.

Moreover, note that our simulations show that the coverage rates are guaranteed by our LR confidence intervals, and that the fuzziness of the ML-estimators do not influence the coverage rates of the calculated fuzzy confidence intervals.

4. Empirical analysis

4.1 SHARE Data and variables

The SHARE Project intends to survey people all over Europe aged 50 or more. It is a long-term panel devoted principally to questions of health of the population. Countries are one dimension of the panel. A complete panel data set could be built from all the modules across the waves. As our empirical application is more to illustrate our method than to conduct a thorough empirical analysis, we complete our estimations based on the data from the generated easySHARE data set. We do not perform a panel analysis. We simply choose to compare Switzerland with its neighbouring countries, i.e. France, Germany, Italy and Austria. The comparison will be by means of two particular variables, the *Childhood health status* (CHILDHOOD_HEALTH) and the *Self-perceived health* (SPHUS). Table 1 gives the description of the variables. The data from the wave 7, corresponding to year 2017, are used. Some filtering conditions are applied to obtain a complete cases data set. Tables 3 and 4 show the numbers of observations of our dataset of both variables by countries and categories.

The variable CHILDHOOD_HEALTH is the perception, which could have a person aged 50 or over of his childhood health status. The variable SPHUS gives the actual individual perception of his or her own state of health. Figure 1 gives the distribution of the crisp data by countries of both variables.

4.2 Fuzzy modelling

Notice that both variables, CHILDHOOD_HEALTH and SPHUS, are coded on a Likert scale going from 1 (Excellent) to 5 (Poor). Such a coding naturally reflects a fuzziness in the data, which we have to model. Indeed, this so-called linguistic variables can easily be fuzzified in triangular fuzzy numbers. In this study, we model these fuzzy numbers by the membership functions given in Table 2.

4.3 Results

We use the R package `FuzzySTs` (Berkachy and Donzé (2020b)) to produce our results. We begin our analysis by estimating the fuzzy confidence intervals by the ratio method at the confidence level 95% for both variables and for each country. These fuzzy intervals are shown in Figures 2 and 3. Tables 5 and 6 give the lower and upper values of the support and core of the intervals. A first remark is that all constructed fuzzy confidence intervals overlap. A simple look at a pairwise comparison level shows us that some of the overlapping sections could be very small, indicating us the potential rejection of the null hypothesis of the equality of means. It is then important to compute the R statistic as show previously. Table 7 gives the R statistics for pairwise comparisons between Switzerland from one side, and its neighbouring countries from another one.

Let us first consider the test of equality of means of the variable CHILDHOOD_HEALTH for Switzerland and Austria, we obtain a very low R statistic of 0.0102735. In this case, we tend to strongly not reject the hypothesis. To confirm this decision, we could run a Fuzzy ANOVA

(FANOVA) for this case (see Berkachy and Donzé (2018a,b)). The FANOVA results for this test are given in Table 8, and show that we cannot reject the hypothesis at 5%.

We consider also the test of equality of means of the variable SPHUS for Switzerland and Germany. In this case, we obtain an R statistic about 0.2470543. This value is relatively far from the value 0. In this sense, we intend to reject the null hypothesis of equality of means of self-perceived health between Switzerland and Germany. The FANOVA results in Table 8 strongly confirms this rejection of hypothesis at 5%. For all other cases, the overlapping regions between intervals seem smaller producing higher R statistics. In these latter cases, we tend to reject the null hypotheses and in fact, these decisions are confirmed by a FANOVA as seen in Table 8.

In conclusion, as shown by this simple analysis, we can infer that Switzerland differs from its neighbouring countries in the perception of *childhood health* and *self-perceived health* as collected by the SHARE survey. This was done by taking into account the fuzziness that is present in the perception of respondents. Thus, being able to treat such vague data and to test it on the equality of means between different groups is a benefit.

5. Conclusion

We have shown how to construct fuzzy confidence intervals by the so-called likelihood ratio method. It appears that such intervals can be very useful in statistical inference. In particular, we propose to use such intervals in the testing of the equality of means of two groups. A decision rule is proposed. We applied successfully this approach on the SHARE data in order to compare Switzerland with its neighbouring countries on the two variables CHILDHOOD_HEALTH and SPHUS. The decisions obtained from our hypothesis testing procedure is confirmed with a fuzzy analysis of variance. It is clear to see that such testing tool is then a absolute asset when data are considered fuzzy.

APPENDIX

Figures

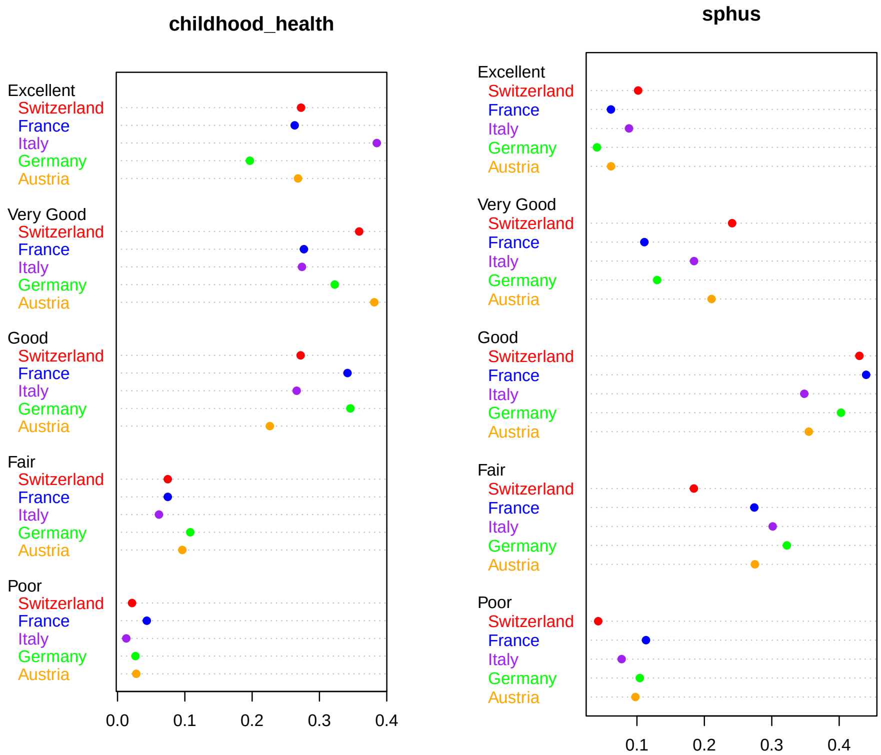

Figure 1: Distributions of the crisp data by countries

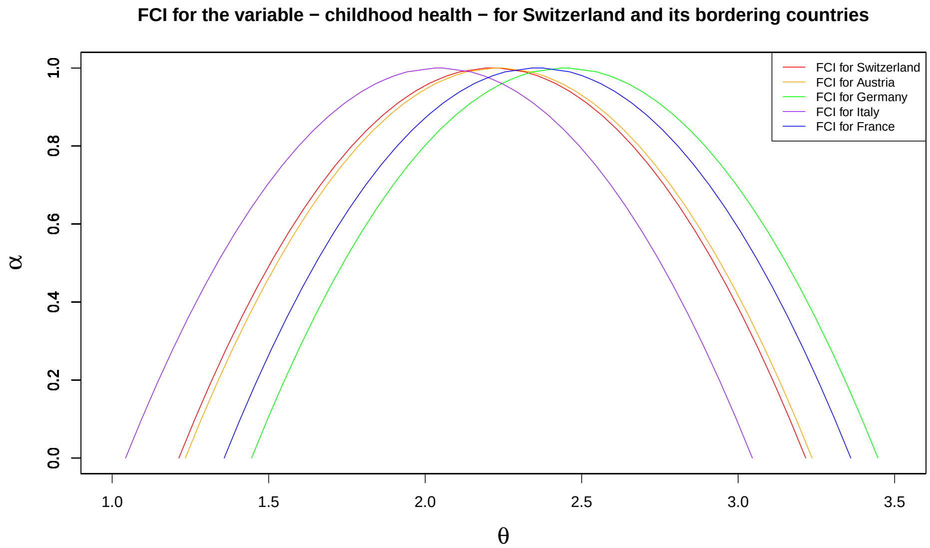

Figure 2: Fuzzy confidence intervals for the mean (CHILDHOOD_HEALTH)

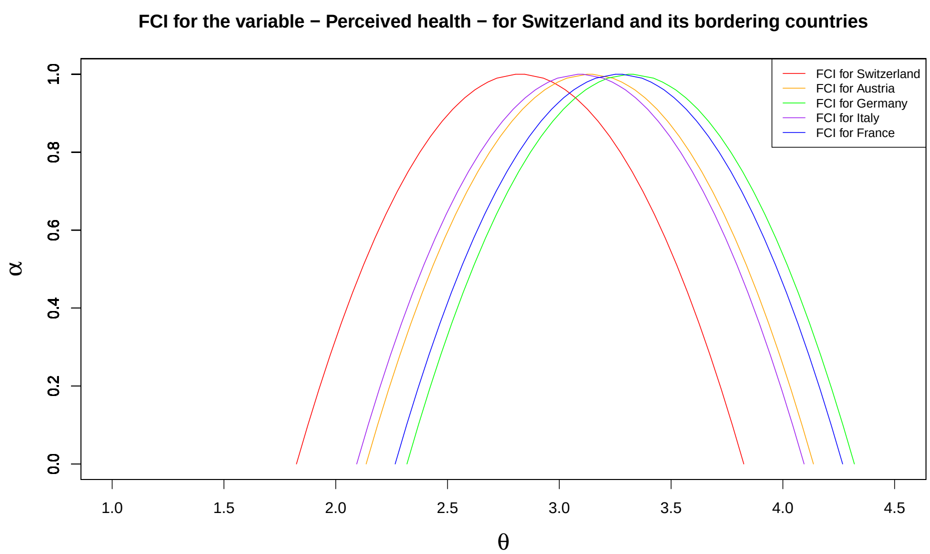

Figure 3: Fuzzy confidence intervals for the mean (SPHUS)

Tables

Variables	Description
CHILDHOOD_HEALTH	Childhood health status (On a scale of 1 to 5; 1: Excellent; 5: Poor)
SPHUS	Self-perceived health (US version) (On a scale of 1 to 5; 1: Excellent; 5: Poor)

Table 1: Variables

Values	$\mu_{\tilde{X}}$
1: Excellent	(0,1,2)
2: Very good	(1,2,3)
3: Good	(2,3,4)
4: Fair	(3,4,5)
5: Poor	(4,5,6)

Table 2: Triangular symmetrical membership functions

	Excellent	Very Good	Good	Fair	Poor	Sum
Austria	705	1003	595	253	73	2629
Germany	575	944	1012	316	78	2925
Italy	1126	801	778	180	38	2923
France	557	586	723	158	92	2116
Switzerland	442	582	441	121	35	1621
Sum	3405	3916	3549	1028	316	12214

Table 3: Number of observations by countries and categories (CHILDHOOD_HEALTH)

	Excellent	Very Good	Good	Fair	Poor	Sum
Austria	162	554	933	723	257	2629
Germany	119	380	1178	943	305	2925
Italy	258	540	1018	881	226	2923
France	130	235	931	580	240	2116
Switzerland	165	391	697	299	69	1621
Sum	834	2100	4757	3426	1097	12214

Table 4: Number of observations by countries and categories (SPHUS)

	Support lower	Core lower	Core upper	Support upper
Austria	1.233616	2.221609	2.247392	3.236316
Germany	1.445193	2.434300	2.457100	3.446688
Italy	1.042829	2.032189	2.055009	3.045396
France	1.357827	2.342394	2.374933	3.359892
Switzerland	1.212950	2.193807	2.234750	3.215834

Table 5: Support and core sets of the fuzzy confidence intervals at 95% (CHILDHOOD_HEALTH)

	Support lower	Core lower	Core upper	Support upper
Austria	2.135964	3.123563	3.149393	4.136868
Germany	2.318464	3.308256	3.330615	4.319930
Italy	2.093722	3.082858	3.106294	4.095051
France	2.265794	3.250962	3.282500	4.267397
Switzerland	1.824227	2.804403	2.845128	3.825295

Table 6: Support and core sets of the fuzzy confidence intervals at 95% (SPHUS)

Variables	Comparison	$d_{\mathrm{SGD}}^{\theta^*}$	$d_{\mathrm{SGD\ opt}}^{\theta^*}$	R
CHILDHOOD_HEALTH	CH-AU	0.020576	2.002791	0.0102735
	CH-IT	0.170280	2.002724	0.0850243
	CH-FR	0.144450	2.002457	0.0721366
	CH-GE	0.231527	2.002167	0.1156383
SPHUS	CH-AU	0.311652	2.000983	0.1557495
	CH-IT	0.269612	2.001186	0.1347265
	CH-FR	0.441819	2.001320	0.2207637
	CH-GE	0.494417	2.001249	0.2470543

Table 7: Distances between FCIs, and R statistics

CHILDHOOD_HEALTH					
	Df	Sum Sq	Mean Sq	F value	Pr(>F)
CH-AU	1	0.42061	0.42061	0.40303	0.52556
CH-IT	1	30.25641	30.25641	30.07949	0
CH-FR	1	19.23801	19.23801	17.6259	3e-05
CH-GE	1	56.14826	56.14826	55.69841	0
SPHUS					
	Df	Sum Sq	Mean Sq	F value	Pr(>F)
CH-AU	1	97.45623	97.45623	92.40485	0
CH-IT	1	75.99624	75.99624	70.34347	0
CH-FR	1	179.4899	179.4899	180.1952	0
CH-GE	1	255.4123	255.4123	270.1339	0

Table 8: Fuzzy ANOVA for the variables CHILDHOOD_HEALTH and SPHUS for different pairwise countries as factor

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